

Aeroelastic stability analysis via multiparameter eigenvalue problems

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Abstract: This paper presents a new method of identifying and analysing stability boundaries in parametric systems using multiparameter spectral theory. Considering our driving application, the analysis of aeroelastic flutter instability, we identify methods by which the location of the stability boundary be expressed as a multiparameter eigenvalue problem and thus solved. This approach yields far-reaching results, including direct solvers for arbitrarily large polynomial problems, iterative and approximate direct solvers for systems that are strongly nonlinear in the frequency domain, and a novel method of system visualisation. These solvers and methods are tested on two aeroelastic section models and the Goland wing benchmark model, and their advantages and limitations are explored.

1. Introduction

The understanding and prediction of aeroelastic instability is a primary concern in the discipline of aeroelasticity. Aeroelastic instability, often termed *flutter* when occurring dynamically, be observed in a wide variety of systems – not only wings and aerofoils, but wall plates [1], hosepipes [2] and more. In a linear system, or the linearisation of a nonlinear system, the onset of flutter can be described by the modal stability criterion:

$$\text{Im}(\chi) > 0 \text{ for stability,} \quad (1)$$

where χ are the time-eigenvalues of the system, transformed according to $q(t) = \hat{q}e^{i\chi t}$ for the system coordinate q [3]. Note that other transforms and nondimensional eigenvalue definitions are possible. A flutter point then be described as a tuple of the modal frequency of instability, $\chi_f \in \mathbb{R}$, and any relevant system parameters (in particular, a local airspeed). As flutter is often associated with structural failure, only the first few flutter points are usually of industrial relevance.

However, even in a linear or linearised system, Eq. 1 is not the only stability criterion available; it corresponds to what is known as the p-method [4]; and a variety of other aeroelastic ‘methods’ are available: A major strain of variants includes the k-method and p-k method, which utilize a structural damping term to describe stability. They are detailed and discussed in a number of reference works [3–5]. In recent years several authors have refined these methods [6–8] and devised new methods. The μ -type methods, including the μ -method by Lind and Brenner [9] and the μ -k method by Borglund [10,11], facilitate the propagation of uncertainty distributions through the system. Irani and

Sazesh [12] characterized flutter instability using stochastic methods, and Afolabi [13,14] applied eigenvector orthogonality conditions from catastrophe theory.

All of these approaches, however, are based on the single-parameter approach of computing a stability metric ($\text{Im}(\chi)$, μ or whatever else) across a range of system parameter values and identifying relevant stability boundaries. We propose an entirely different method of analysis. We show that the solution of an aeroelastic system for its flutter points – or the analysis of any other frequency-domain stability problem – is nothing other than a multiparameter eigenvalue problem. We will demonstrate how this approach leads to a number of improved solvers for a wide range of parametric stability problems drawn from the field of aeroelasticity. Our methods are equally applicable in other fields.

2. Multiparameter analysis

Consider a linear finite-dimensional system with eigenvector $\mathbf{x} \in \mathbb{C}^n$, continuously dependent on both an eigenvalue parameter $\chi \in \mathbb{C}$, and another structural or environmental parameter $p \in \mathbb{R}$:

$$A(\chi, p)\mathbf{x} = \mathbf{0}, \quad (2)$$

where $A \in \mathbb{C}^{n \times n}$. Any complex-valued structural parameter can be split into two real parameters. We then note that the condition for the stability boundary, $\text{Im}(\chi) = 0$, is equivalent to defining the problem with $\chi \in \mathbb{R}$. However, under $\chi \in \mathbb{R}$ a solution to Eq. 2 only exists on the stability boundary, and nowhere else. To define some form of solution in the subcritical and supercritical areas (above and below the stability boundary, respectively), following [15], we take the complex conjugate of Eq. 2 as another equation:

$$A(\chi, p)\mathbf{x} = \mathbf{0}, \quad (3)$$

$$\bar{A}(\chi, p)\bar{\mathbf{x}} = \mathbf{0}. \quad (4)$$

As $p \in \mathbb{R}$ and $\chi \in \mathbb{R}$ are unaffected by conjugation, this operation enforces these conditions. This procedure has been utilized before in the analysis of delay differential equations [15], and (in a limited form) in the context of Hopf bifurcation prediction [16], though in the latter its significance appears not to have been recognised. Equation 3 a multiparameter eigenvalue problem (MEP): an eigenvalue problem in which the eigenvalue point is not simply defined by a scalar and an eigenvector, but by an n -tuple and an eigenvector. A number of methods of analysis have been developed for such problems, and in this paper we will explore some of these.

3. Linear and polynomial problems

3.1. Direct solution

Consider a linear instability problem:

$$(A + B\chi + Cp)\mathbf{x} = \mathbf{0}, \quad (5)$$

$$(\bar{A} + \bar{B}\chi + \bar{C}p)\bar{\mathbf{x}} = \mathbf{0}. \quad (6)$$

Post-multiplying Eq. 5 by $\bar{\mathbf{C}}\mathbf{y}$ and premultiplying Eq. 6 by $\mathbf{C}\mathbf{x}$, we obtain

$$(A + B\chi + Cp)\mathbf{x} \otimes (\bar{\mathbf{C}}\mathbf{y}) = \mathbf{0}, \quad (7)$$

$$(\mathbf{C}\mathbf{x}) \otimes (\bar{A} + \bar{B}\chi + \bar{C}p)\mathbf{y} = \mathbf{0}. \quad (8)$$

Equations 7 and 8 are equal to zero and so we equate them. After cancelling the terms in p , the result becomes:

$$\Delta_1 \mathbf{z} = \chi \Delta_0 \mathbf{z}, \quad (9)$$

with an enlarged eigenvector $\mathbf{z} = \mathbf{x} \otimes \mathbf{y}$ and the operator determinants

$$\Delta_0 = B \otimes \bar{C} - C \otimes \bar{B}, \quad (10)$$

$$\Delta_1 = C \otimes \bar{A} - A \otimes \bar{C}, \quad (11)$$

$$\Delta_2 = A \otimes \bar{B} - B \otimes \bar{A}, \quad (12)$$

which are of size n^2 relative to system coefficients of size n . Equation 9 is a generalized eigenvalue problem (GEP), in the single parameter χ . GEP solvers are very widely available.

The operator determinants also define a GEP in p . Multiplying by $\bar{\mathbf{B}}\mathbf{y}$ and $\mathbf{B}\mathbf{x}$, we have:

$$\Delta_2 \mathbf{z} = p \Delta_0 \mathbf{z}. \quad (13)$$

However, only one of Eq. 9 or Eq. 13 need be solved: the solutions of one can be substituted back into the original system, which yields smaller GEP for the other parameter. Alternatively, Rayleigh quotients be used. The problem's stability boundary has thus been computed directly. This solution method is known as the operator determinant method. Its computational complexity is $\mathcal{O}(n^6)$ [17–19]; solving the GEP via the QZ algorithm (an $\mathcal{O}(n^3)$ process) [20], with operator determinants of size n^2 . The operator determinant method has not previously been used in aeroelasticity, and has only rarely seen engineering application in the study of dynamic model updating [21]. We note that a variety of other iterative methods are also available for the solution of linear MEPs, including the Jacobi-Davidson [22], Implicitly Restarted Arnoldi [23] and Harmonic Rayleigh-Ritz [24] methods.

3.2. Linearisation of polynomials

Any polynomial MEP can be linearised [18,25]; a process which resembles the well-known linearisation of single-parameter problems. For example, a quadratic problem $(A + B\chi + C\tau + D\chi p + E\chi^2 + Fp^2)\mathbf{x} = \mathbf{0}$ be linearised with the eigenvector definition $\mathbf{q} = [\mathbf{x}; \chi\mathbf{x}; p\mathbf{x}]$.

$$\left(\begin{bmatrix} A & B & C \\ 0 & -I_n & 0 \\ 0 & 0 & -I_n \end{bmatrix} + \begin{bmatrix} 0 & D & E \\ I_n & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \chi + \begin{bmatrix} 0 & 0 & F \\ 0 & 0 & 0 \\ I_n & 0 & 0 \end{bmatrix} p \right) \begin{bmatrix} \mathbf{x} \\ \chi\mathbf{x} \\ p\mathbf{x} \end{bmatrix} = \mathbf{0}. \quad (14)$$

Quadratic problems are particularly relevant in aeroelasticity given the near-quadratic dependence of most systems on airspeed and modal frequency. There is also an alternate method of linearisation, known as quasilinearisation [18], which increases the number of eigenvalue parameters instead of the coefficient size. In this brief work however we focus on standard linearisation.

3.3. Singularity

A linear MEP can be singular; as governed by the singularity of Δ_0 . When this occurs the operator determinant method as described breaks down [18,26]. A number of problems that arise in the study of aeroelasticity are singular, because the linearization of polynomial problems tends to generate singular linear problems, even if all the coefficients of the original problem are at full rank (cf. Eq. 14). Recently, an extension to the operator determinant method was proposed that allows it to cope with this singularity. Muhič and Plestenjak [25] proved that the eigenvalues of a polynomial system are equivalent to the finite regular eigenvalues of the pair of singular operator determinant GEPs constructed via linearization. The finite regular eigenvalues of Eq. 5 and 6 are the pairs (χ, p) such that [26]:

$$\text{rank}(A + B\chi + Cp) < \max_{(s,t) \in \mathbb{C}^2} \text{rank}(A + Bs + Ct), \quad (15)$$

that is, they are the points that cause the singular problem to have its maximum rank. On the basis of this proof, Muhič and Plestenjak [25] devised a set of algorithms which would extract the common regular part of the singular matrix pencils $\Delta_1 - \chi\Delta_0$ and $\Delta_2 - p\Delta_0$. This common regular part is represented by two smaller nonsingular matrix pencils $(\Delta_{1\text{ns}} - \chi\Delta_{0\text{ns}}$ and $\Delta_{2\text{ns}} - p\Delta_{0\text{ns}})$, which can be solved by GEP solvers as per normal. The algorithms involved in the extraction of the common regular part are presented in [25] and published also in code [27]. Only one of the additional iterative algorithms mentioned in Section 3.1 are capable of solving singular systems with this extension; this is the Jacobi-Davidson method [22]. A comparison of this method with the operator determinant method for singular systems indicated that the latter is more computationally efficient [28].

4. Nonlinear problems

4.1. Direct methods

A variety of nonlinear eigenvalue problems arise in aeroelasticity, and take a variety of forms. Note that such problems are not equivalent to nonlinear stability problems; being already in the frequency domain. One particularly common class are polynomial problems containing a nonlinear scalar function – in aeroelasticity often Theodorsen’s function [4]. Such problems be transformed into approximate polynomial problems (and thus solved) with the choice of an appropriate approximation for the nonlinear function. Polynomial, rational or rational fractional-order approximations are all admissible. We give a specific example of this method in Section 5.

4.2. Iterative methods

Another more general approach to nonlinear MEPs is the use of iterative algorithms that assume nothing about the problem’s internal structure. Ruhe [29] proposed a method of successive linear problems for one-parameter eigenvalue problems; and generalizations to this method for MEPs were published independently by Pons [30] and Plestenjak [31]. For the system of Eq. 2-3 taking first-order Taylor series in the eigenvalue variables, we obtain an implicit fixed-point iteration via a linear MEP:

$$(A_k + \Delta\chi_k X_k + \Delta p_k P_k)\mathbf{x} = 0, \quad (16)$$

$$(\bar{A}_k + \Delta\chi_k \bar{X}_k + \Delta p_k \bar{P}_k)\bar{\mathbf{x}} = 0, \quad (17)$$

where $A_k = A(\chi_k, p_k)$, $P_k = \partial_p A(\chi_k, p_k)$, $X_k = \partial_\chi A(\chi_k, p_k)$ and $\Delta\chi_k = \chi_{k+1} - \chi_k$, etc. This linear problem be solved at each step with the operator determinant method. This however comes at the cost of $\mathcal{O}(n^6)$ computational complexity [31].

Alternatively, a more computationally efficient method of solving nonlinear MEPs be devised by applying Newton’s method to the determinant of the nonlinear matrix coefficient. Defining a state vector $\mathbf{v} = [\chi, p]^T$ and the complex-valued scalar determinant function $z = \det(A(\mathbf{v}))$, we obtain the Newton iteration

$$\mathbf{v}_{k+1} = \mathbf{v}_k - J(\mathbf{v}_k)^{-1} \mathbf{F}(\mathbf{v}_k), \quad (18)$$

with a real-valued residual function

$$\mathbf{F}(\mathbf{v}) = \begin{bmatrix} \text{Re}(z(\mathbf{v})) \\ \text{Im}(z(\mathbf{v})) \end{bmatrix} = \mathbf{0}, \quad (19)$$

and where J is the Jacobian matrix of \mathbf{F} with respect to \mathbf{v} . We term this method the iterated contour plot (ICP) as it can be related to an iterative formulation of the contour plot [30]. It has been applied

(in basic form) to two-parameter linear MEPs by Podlevskii [32,33] and to nonlinear MEPs independently by Pons [30] and Plestenjak [31]. This method has computational complexity $\mathcal{O}(n^3)$, for LU-based determinant evaluation [34].

4.3. The contour plot

Modal damping or root locus plots are traditional methods of visualising the stability behaviour of an aeroelastic system. However, neither is suitable for visualising our multiparameter formulations, as we have $\chi \in \mathbb{R}$ always. To this purpose we introduce the contour plot, as per Pons and Gutschmidt [30,35]. This involves plotting contours of $\text{Re}(z)$ and $\text{Im}(z)$, where $z = \det(A(\chi, p))$ for Eq. 2; i.e. the real and imaginary parts of the matrix function determinant, as a function of its parameters. These contours be plotted by evaluating z over a grid of χ and p ; their intersection represents a point $z = 0$, i.e. a stability boundary. This process is particularly useful for strongly nonlinear matrix functions, including nondifferentiable ones. A variety of contour plots are presented in Section 5.

5. Numerical experiments

5.1. Section model

As an initial test system we consider an aerofoil with two degrees of freedom (plunge h and twist θ). Figure 1 shows a schematic of such a system, with dimensionless parameters as per Table 1 and the airspeed parameter Y ; the airspeed per semichord. A frequency domain analysis of this system, under Theodorsen's unsteady aerodynamic theory yields a problem of the form:

$$\left(\left(G_0 + G_1 \frac{1}{\kappa} + G_2 \frac{C(\kappa)}{\kappa} + G_3 \frac{C(\kappa)}{\kappa^2} \right) \chi^2 - D_0 \chi - K_0 \right) \mathbf{x} = \mathbf{0}, \quad (20)$$

where $\mathbf{x} = [h; \theta]$, $\kappa = Y/\chi$ is the reduced frequency, and $C(\kappa)$ is Theodorsen's function, composed of a number of Bessel functions [4]. The matrix coefficients in Eq. 20 are:

$$G_0 = \frac{1}{\mu} \begin{bmatrix} 2 & -a - r_\theta \\ -a - r_\theta & \frac{1}{8} + a^2 + r^2 \end{bmatrix}, G_1 = \frac{1}{\mu} \begin{bmatrix} 0 & -1 \\ 0 & -\frac{1}{2} + a \end{bmatrix}, G_2 = \frac{1}{\mu} \begin{bmatrix} -2 & -1 + 2a \\ 1 + 2a & \frac{1}{2} - 2a^2 \end{bmatrix} \quad (21)$$

$$G_3 = \frac{1}{\mu} \begin{bmatrix} 0 & -2 \\ 0 & 1 + 2a \end{bmatrix}, D_0 = 2t \begin{bmatrix} \zeta_h \omega_h & 0 \\ 0 & r^2 \zeta_\theta \omega_\theta \end{bmatrix}, K_0 = \begin{bmatrix} \omega_h^2 & 0 \\ 0 & r^2 \omega_\theta^2 \end{bmatrix} \quad (22)$$

with parameters as per Table 1. See Pons and Gutschmidt [28] or Hodges and Pierce [4] for details. Taking $C(\kappa) = 1$ corresponds to the assumption of quasisteady aerodynamics, and with a change of variables produces a polynomial system:

$$(G_0 \chi^2 + G_1 Y \chi + G_2 Y^2 - D_0 \chi - K_0) \mathbf{x} = \mathbf{0}, \quad (23)$$

where $Y = U/b$ is the local airspeed per semichord. This polynomial system be linearized and solved with the operator determinant method of Section 3.

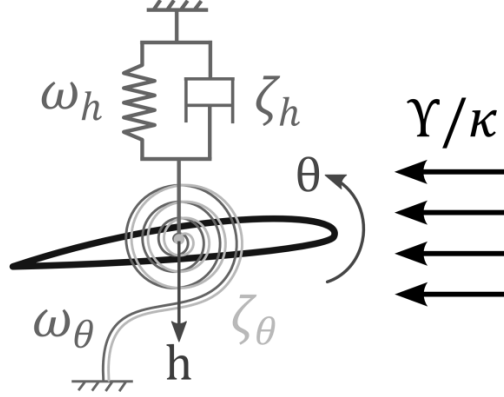


Figure 1. Schematic of section model

Table 1: Parameter values for the section model

Parameter	Value
mass ratio – μ	20
radius of gyration – r	0.4899
bending nat. freq. – ω_h	0.5642 rad/s
torsional nat. freq. – ω_θ	1.4105 rad/s
bending damping – ζ_h	1.4105 %
torsional damping – ζ_θ	2.3508 %
static imbalance – r_θ	–0.1
pivot point location – a	–0.2

The results of this process are shown in Figure 2(a), which includes a contour plot of the system. The flutter point is located at $\chi = 1.20$ rad/s and $Y = 1.98$ Hz ($\kappa = 0.606$). This agrees with nondimensional analytical results by Hodges and Pierce [4]. We can, however, go further than an analytical approach: we increase the matrix coefficient system size arbitrarily (and the polynomial system order) and still obtain exact solutions. A direct solver for polynomial flutter problems of arbitrary size and order has never before been presented.

We can also consider the case when $C(\kappa)$ is fully variable. The resulting MEP is nonlinear; however a variety of approximations for Theodorsen's function are available. We take a rational function given by Jones [36]:

$$C(\kappa) = \frac{\frac{1}{2}\kappa^2 + c_1\kappa + c_2}{\kappa^2 + c_3\kappa + c_2}, \quad (23)$$

with $c_1 = -0.2808t$, $c_2 = -0.01365$, $c_3 = -0.3455t$. Manipulating Eq. 20 we then obtain a polynomial problem of maximum order $\kappa^4\chi^2$, requiring a custom linearization of 10 blocks width. This be solved via the operator determinant method in under 0.2s on a laptop computer. The results are shown in Figure 2(b), also with a contour plot of the system. The fact that this solver is direct is a significant advantage over existing solvers for systems of this form.

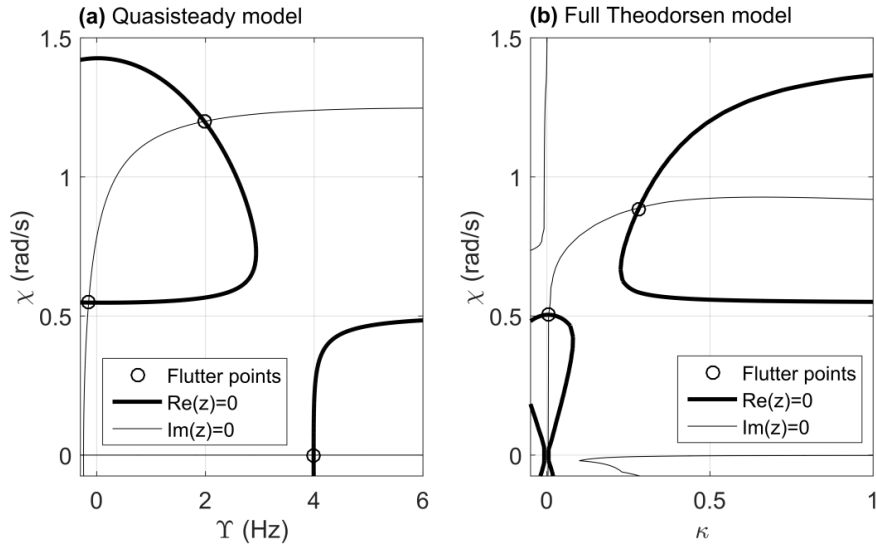


Figure 2. Flutter point results for the section model with two aerodynamic models.

5.2. Goland wing

As a benchmark test for our iterative algorithms, we analyse the well-known Goland Wing test case – representing a cantilever Euler-Bernoulli beam and Saint-Venant torsion model, with strip theory Theodorsen aerodynamics [37]. Originally a differential MEP (containing spatial derivatives as well as eigenvalue parameters), it is transformed by the Generalised Laplace Transform Method (GLTM) [35] into a nonlinear algebraic problem of fixed size (12×12). This transformation is without discretisation error, though it comes at the cost of obscuring the internal structure of the model – hence we treat the transformed problem as black-box nonlinear MEP. There is a small variation in parameter values for the Goland wing and so we take parameter values from Pons and Gutschmidt [35] and Wang [38]. For these parameters the Goland wing’s first flutter point is located at airspeed $U_F = 138$ m/s and modal frequency $\chi_F = 69.9$ rad/s. The first divergence point (static instability) is located nearby at $U_D = 253$ m/s. Figure 3 shows example SLP iteration paths converging to the flutter point, divergence point, and undamped modal frequencies at zero airspeed (also technically flutter points). All iterations are convergent, and agree with the results from the

literature. Figure 4 shows the convergence basins of the SLP and ICP algorithms to the flutter point, computed numerically. The SLP algorithm has the larger basin; though both are very satisfactory and the ICP is more computationally efficient.. The SLP algorithm is likely to be attractive for smaller systems with little a priori knowledge, whereas the ICP is effective for larger and more expensive systems, for which an initial flutter point estimate from an approximate model be available.

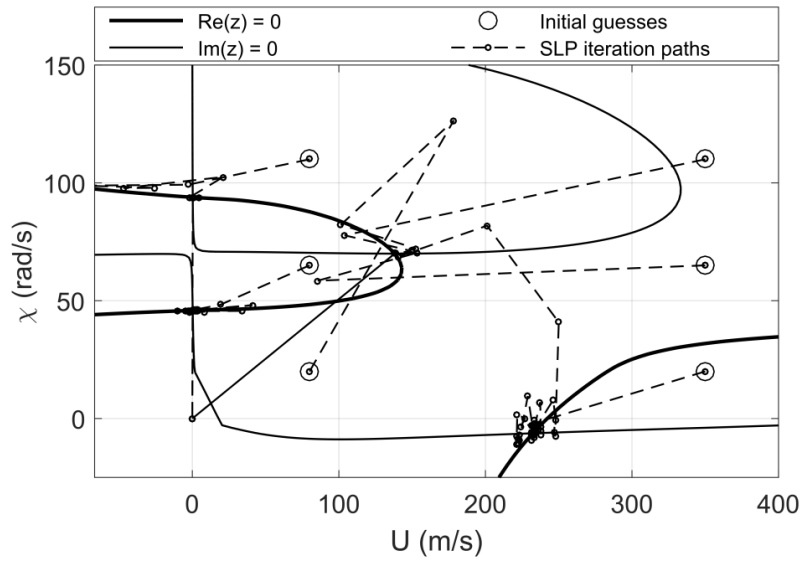


Figure 3. Six example iterations of the SLP algorithms applied to the Goland wing.

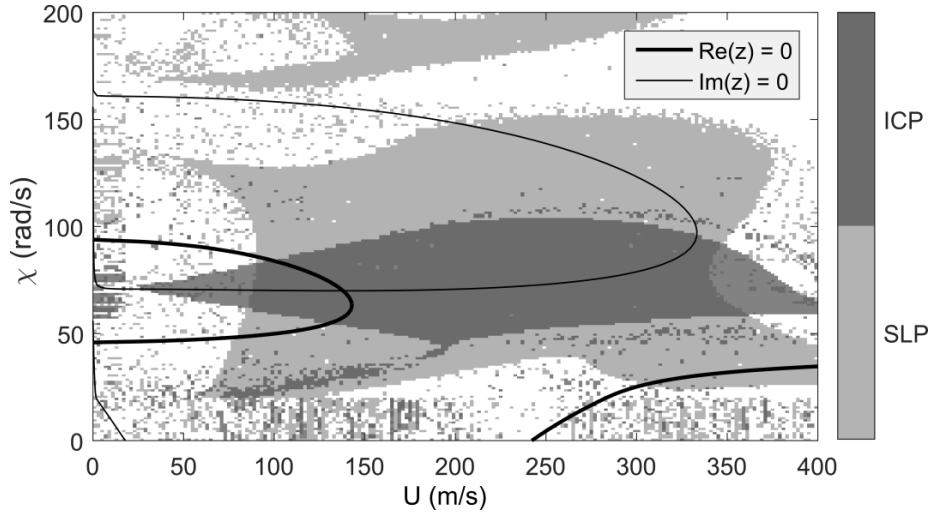


Figure 4. Convergence basins of the SLP and ICP algorithms to the Goland wing first flutter point.

6. Conclusions

In this paper we have demonstrated and discussed the use of multiparameter solution techniques for the solution of aeroelastic stability problems. We have introduced the link between multiparameter spectral theory and stability analysis, and we showed how this link can be used to reformulate stability problems with a complex-valued stability metric and a pertinent environmental parameter into a two-parameter eigenvalue problem. We demonstrated that this allows the direct solution of polynomial stability problems, as well as approximate direct and iterative solution methods for strongly nonlinear problems. The application of multiparameter methods – in aeroelasticity and in other disciplines – has the potential to provide a wide variety of new methods for stability analysis.

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